

# Bounded orbits for photons as a consequence of extra dimensions

P. A. González\* and Marco Olivares†

*Facultad de Ingeniería, Universidad Diego Portales,  
Avenida Ejército Libertador 441, Casilla 298-V, Santiago, Chile.*

Yerko Vásquez‡

*Departamento de Física, Facultad de Ciencias, Universidad de La Serena,  
Avenida Cisternas 1200, La Serena, Chile.*

(Dated: November 26, 2015)

## Abstract

In this work, we study the geodesic structure for a geometry described by a spherically symmetric four-dimensional solution embedded into a five-dimensional space known as brane-based spherically symmetric solution. Mainly, we have found that the extra dimension contributes to the particle test mass, that we call *the anomalous test mass*, which is positive for photons. This fact implies that there are bounded orbits for the photons, like planetary and circular stable orbits that have not been observed for other geometries.

---

\* pablo.gonzalez@udp.cl

† marco.olivaresr@mail.udp.cl

‡ yvasquez@userena.cl

## CONTENTS

I. Introduction	3
II. Geodesics	4
III. Circular orbits	6
IV. Orbits	8
A. Bounded orbits	8
1. Planetary orbit	8
B. Unbounded orbits	10
1. Deflection of light	10
V. Radial trajectories	12
A. Bounded trajectories:	13
B. Unbounded trajectories:	14
VI. Summary	14
Acknowledgments	15
References	15

## I. INTRODUCTION

Extra-dimensional gravity theories have a long history that begins with an original idea propounded by Kaluza and Klein [1] as a way for unifying the electromagnetic and gravitational fields, and nowadays finds a new realization within the modern string theory [2, 3]. Among the higher-dimensional models of gravity, the five-dimensional Randall-Sundrum brane worlds models [4, 5] have acquired great attention in the last decade. In particular, their second model [5] can be described by a single brane, i.e a 3+1-dimensional hypersurface embedded into a higher-dimensional space-time, in which the matter and gauge fields of the standard model are confined to it, while the gravitational field can propagate in the fifth dimension, which have an infinite size. From the cosmological point of view, brane worlds offer a novel approach to our understanding of the evolution of the universe and they have exhibited interesting cosmological implications, as witnessed in the study of missing matter problems, as well as providing a new mechanism to explain the acceleration of the Universe based on modifications of general relativity (GR), instead of introducing an exotic content of matter, for instance see Ref. [6] and the references therein. On the other hand, Dvali-Gabadadze-Porrati (DGP) brane world model [7], whose gravity behaves as four-dimensional at short distance scale but it shows higher-dimensional nature at larger distances, also has attracted a great interest in the last decade [8–10]. This brane world model is characterized by the brane on which the fields of the standard model are confined, and contains the induced Einstein-Hilbert term. It also exhibits several cosmological features [11–14]. Also, it was shown that the effective 4-dimensional equations obtained by projecting the 5-dimensional metric onto the brane acquire corrections terms to GR [15]. Additionally, brane world models with a non-singular or thick brane have been considered in the literature, see for instance [16] and references there in.

In this work, we study the geodesic structure for a geometry described by a Schwarzschild four-dimensional solution embedded into a five-dimensional space. The metric is given by

$$ds^2 = - \left( 1 - \frac{2m}{\tilde{r}} \right) (d\tilde{x}^4)^2 + \frac{d\tilde{r}^2}{1 - \frac{2m}{\tilde{r}}} + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 \sin^2 \tilde{\theta} d\tilde{\phi}^2 + (d\tilde{x}^5)^2, \quad (1)$$

where  $\tilde{x}^5$  stands for the extra dimension. Besides, this line element can be related to the so-called black-string [17]. Furthermore, by adding an extra flat dimension to the Kerr solution of GR and performing a boost in the fifth dimension to this five-dimensional metric,

and then compactifying the extra dimension, a new four-dimensional charged spherically symmetric black hole together with a Maxwell and dilaton field was obtained in Ref. [18], the hidden symmetries and geodesics of Kerr spacetime in Kaluza-Klein theory was studied in Ref. [19]. In Ref. [20], the authors set up the mass parameter to be far below the value necessary for a black-hole solution, and investigate how the embedding of the solution into the extra-dimensional scenario can be related to the classic tests performed in GR such as the perihelion shift of the planet Mercury, the deflection of light by the Sun, and the gravitational redshift of atomic spectral lines. In this paper we adopt the same constraint on the mass parameter. It is worth to mention that the classical tests of GR have been examined for various spherically symmetric static vacuum solutions of braneworld models, for instance see [21]. On the other hand, it is worth to mention that the geometrical localization mechanism implies a four-dimensional mass for the photon [22].

This paper is organized as follow: in Sec. II we present the procedure to obtain the equations of motion for neutral particles in the brane-based spherically symmetric solutions. Then, in Sec. III we give the exact solution for the circular orbits. The analytical solution for the orbits with angular momentum in terms of the the  $\wp$ -Weierstrass elliptic function is described in Sec. IV. Then, in Sec. V, we study the radial trajectories. Finally, in Sec. VI we conclude with some comments and final remarks.

## II. GEODESICS

First, we consider a light-cone type transformation in spherical coordinates, embedded into a five-dimensional space [20] given by

$$r = \tilde{r}, \quad \theta = \tilde{\theta}, \quad \phi = \tilde{\phi}, \quad x^4 = \frac{\tilde{x}^4 + \tilde{x}^5}{\sqrt{2}}, \quad x^5 = \frac{\tilde{x}^4 - \tilde{x}^5}{\sqrt{2}}. \quad (2)$$

In this way, Eq. (1) becomes

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 + \frac{m}{r} (dx^4)^2 + \frac{m}{r} (dx^5)^2 + 2 \left( \frac{m}{r} - 1 \right) dx^4 dx^5. \quad (3)$$

Now, with the aim to study the motion of neutral particles around a brane-based spherically symmetric solution, we derive the geodesic equations. So, the Lagrangian associated to the metric (3) results to be

$$2\mathcal{L} = \frac{\dot{r}^2}{1 - \frac{2m}{r}} + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 + \frac{m}{r} (\dot{x}^4)^2 + \frac{m}{r} (\dot{x}^5)^2 + 2 \left( \frac{m}{r} - 1 \right) \dot{x}^4 \dot{x}^5 = -\mu^2, \quad (4)$$

where  $\dot{a} = da/d\tau$ ,  $\tau$  is an affine parameter along the geodesic that we choose as the proper time, and  $\mu$  is the test mass of the particle ( $\mu = 1$  for massive particles and  $\mu = 0$  for photons).

The equations of motion are obtained from  $\dot{\Pi}_q - \frac{\partial \mathcal{L}}{\partial q} = 0$ , where  $\Pi_q = \partial \mathcal{L} / \partial \dot{q}$  are the conjugate momenta to the coordinate  $q$ , which yields

$$\dot{\Pi}_r = -\frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-2} \dot{r}^2 + r(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - \frac{m}{2r^2} (\dot{x}^4 + \dot{x}^5)^2, \quad (5)$$

$$\dot{\Pi}_\theta = r^2 \sin \theta \cos \theta \dot{\phi}^2, \quad \dot{\Pi}_\phi = 0, \quad \dot{\Pi}_{x^4} = 0 \quad \text{and} \quad \dot{\Pi}_{x^5} = 0. \quad (6)$$

So, we can observe that the conjugate momenta associated to the coordinates  $\phi$ ,  $x^4$ , and  $x^5$  are conserved. Therefore

$$\Pi_r = \frac{\dot{r}}{1 - \frac{2m}{r}}, \quad \Pi_\theta = r^2 \dot{\theta}, \quad \Pi_\phi = r^2 \sin^2 \theta \dot{\phi} \quad (7)$$

$$\Pi_{x^4} = \frac{m}{r} \dot{x}^4 + \left(\frac{m}{r} - 1\right) \dot{x}^5, \quad \text{and} \quad \Pi_{x^5} = \frac{m}{r} \dot{x}^5 + \left(\frac{m}{r} - 1\right) \dot{x}^4. \quad (8)$$

Now, without lack of generality we consider that the motion is developed in the invariant plane  $\theta = \pi/2$  and  $\dot{\theta} = 0$ . So the above equation can be written as

$$r^2 \dot{\phi} \equiv h, \quad \frac{m}{r} \dot{x}^4 + \left(\frac{m}{r} - 1\right) \dot{x}^5 \equiv c_1 \quad \text{and} \quad \frac{m}{r} \dot{x}^5 + \left(\frac{m}{r} - 1\right) \dot{x}^4 \equiv c_2 \quad (9)$$

where  $h$  is the angular momentum of the particle,  $c_1$  and  $c_2$  are dimensionless integration constants. Thus, by defining  $c_1 + c_2 \equiv -k$  and  $c_1 - c_2 \equiv b$ , we obtain

$$\dot{x}^4 = \frac{1}{2} \left( \frac{k}{1 - \frac{2m}{r}} + b \right) \quad \text{and} \quad \dot{x}^5 = \frac{1}{2} \left( \frac{k}{1 - \frac{2m}{r}} - b \right). \quad (10)$$

Note that the coordinate  $\tilde{x}^5$  can be obtained subtracting Eqns. (10),

$$\dot{\tilde{x}}^5 = \frac{\dot{x}^4 - \dot{x}^5}{\sqrt{2}} = b \implies \tilde{x}^5 = b\tau. \quad (11)$$

Thus, the coordinate  $x^5$  increases linearly with the affine parameter  $\tau$ . Therefore, the particles can scape to the additional dimension. However, this is not a problem due to the fifth dimension is assumed to be compact with a very small radius. Finally, by substituting Eqs. (9) and (10) into Eq. (4), it follows that

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{k^2}{2} - \left(1 - \frac{2m}{r}\right) \left(\mu^2 + \frac{b^2}{2} + \frac{h^2}{r^2}\right). \quad (12)$$

Note that the constant  $b$  of the extra dimension contributes to the particle test mass in the above equation. So, we define the new constant  $\tilde{\mu}^2 \equiv \mu^2 + b^2/2$ , which we may call *the anomalous test mass*. Therefore, the effective potential  $V(r)$  can be written as

$$V(r) = \left(1 - \frac{2m}{r}\right) \left(\tilde{\mu}^2 + \frac{h^2}{r^2}\right), \quad (13)$$

which is showed in Fig. 1, where the point indicates the last stable circular orbit. Observe that for photons  $\mu = 0$ ; however,  $\tilde{\mu}^2 = b^2/2$  is positive. This fact implies that there are bounded orbits for the photons. In the following, despite of the geodesics are for photons and particles, we will plot only the photon case ( $\tilde{\mu}^2 = 0.1 < 1$ ).

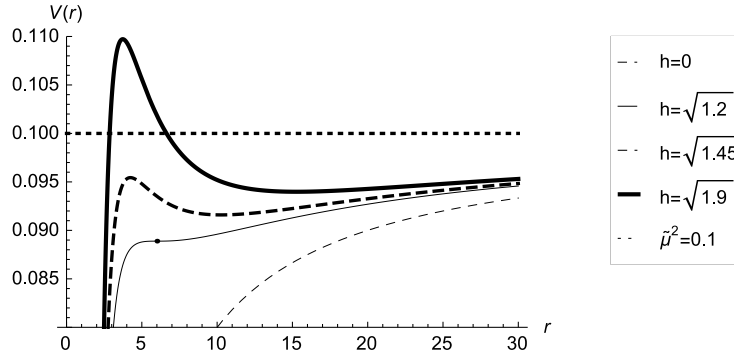


FIG. 1. Effective potential for photons with  $m = 1$  and  $\tilde{\mu}^2 = 0.1$ .

On the other hand, if we consider Eq. (12), the orbit in polar coordinates is given by

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{h^2} \left[ \frac{k^2}{2} - \left(1 - \frac{2m}{r}\right) \left(\tilde{\mu}^2 + \frac{h^2}{r^2}\right) \right]. \quad (14)$$

This equation determines the geometry of the geodesic. Moreover, the orbital Binet equation yields

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} \tilde{\mu}^2 + 3mu^2, \quad (15)$$

where we have used the Keplerian change of variable  $r$  to  $u = 1/r$ .

### III. CIRCULAR ORBITS

It is known that the circular orbits ( $r_{c.o.}$ ) correspond to an extreme value of the potential, that is,  $dV(r)/dr = 0$ . The last stable circular orbit occurs when the angular momentum is  $h_{LSCO} = 2\sqrt{3}m\tilde{\mu}$ , which depends on the anomalous test mass. However, the minimum

radius of stable circular orbit is  $r_{LSCO} = 6m$ , and does not depend on the anomalous test mass. Therefore, if  $h < h_{LSCO}$  there are not circular orbits, and if  $h > h_{LSCO}$  the circular orbits can be stable ( $r_S$ ) and unstable ( $r_U$ ), which yields

$$r_U = \frac{6m h^2}{h_{LSCO}^2} \left( 1 - \sqrt{1 - \frac{h_{LSCO}^2}{h^2}} \right) \quad r_S = \frac{6m h^2}{h_{LSCO}^2} \left( 1 + \sqrt{1 - \frac{h_{LSCO}^2}{h^2}} \right). \quad (16)$$

It is important to mention that, despite of the mass test of the photon is null the anomalous test mass is positive. Therefore, there are stable circular orbits for the light. Now, in order to estimate the radius of the circular orbits for photons, consider for example a circular orbit for photons around the Sun, without considering the effect of other bodies. The geometrical mass  $m$  and the ratio  $b/h$  are constrained to [20]:

$$m = 1.4766250385(1)km, \quad \frac{b}{h} < 2.403015 \times 10^{-8} km^{-1}. \quad (17)$$

Thus, considering  $b/h \approx 2.403015 \times 10^{-8} km^{-1}$  from Eqn. (16) we obtain the radius  $r_S \approx 2.3 \times 10^{15} km$ .

On the other hand, the periods for one complete revolution of these circular orbits, measured in proper time and coordinate time ( $x^4, x^5$ ), are

$$T_\tau = 2\pi \sqrt{\frac{r_{c.o.}^3 - 3mr_{c.o.}^2}{m\tilde{\mu}^2}}, \quad T_4 = 2\pi \sqrt{\frac{r_{c.o.}^3}{2m}} + \frac{b}{2} T_\tau, \quad T_5 = 2\pi \sqrt{\frac{r_{c.o.}^3}{2m}} - \frac{b}{2} T_\tau. \quad (18)$$

On the other hand, expanding the effective potential around to  $r = r_S$ , we can write

$$V(r) = V(r_S) + V'(r_S)(r - r_S) + \frac{1}{2}V''(r_S)(r - r_S)^2 + \dots, \quad (19)$$

where  $'$  means derivative with respect to the radial coordinate. Obviously, in this orbits  $V'(r_S) = 0$ . So, by defining the *smaller* coordinate  $x = r - r_S$ , together with *the epicycle frequency*  $\kappa^2 \equiv V''(r_S)/2$  [23], we can rewrite the above equation as  $V(x) \approx \frac{k_S^2}{2} + \kappa^2 x^2$  where  $\frac{k_S^2}{2}$  is the energy of the particle at the stable circular orbit. Also, it is forward to see that test particles satisfy the harmonic equation of motion  $\ddot{x} = -\kappa^2 x$ . Therefore, in our case, the epicycle frequency is given by

$$\kappa^2 = \frac{m\tilde{\mu}^2}{r_S^3} \left[ \frac{r_S - 6m}{r_S - 3m} \right]. \quad (20)$$

Notice that when  $b \rightarrow 0$ ,  $\kappa \rightarrow \kappa_{Schw}$ , where  $\kappa_{Schw}$  is the epicycle frequency in the Schwarzschild case.

## IV. ORBITS

Now, in order to obtain a full description of the motion of the neutral particles, we will find analytically the geodesics. So, let us rewrite Eq. (14) as

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{(\tilde{\mu}^2 - k^2/2)}{h^2} r P(r), \quad (21)$$

where the characteristic polynomial  $P(r)$  is given by

$$P(r) = -r^3 + \left(\frac{2m\tilde{\mu}^2}{\tilde{\mu}^2 - k^2/2}\right) r^2 - \left(\frac{h^2}{\tilde{\mu}^2 - k^2/2}\right) r + \frac{2mh^2}{\tilde{\mu}^2 - k^2/2}. \quad (22)$$

So, we can see that depending of the nature of its roots, we shall obtain the allowed motions for this configuration. Therefore, by integrating Eq. (21), we obtain the polar form to the first kind orbit of the neutral massive particles, which yields

$$r(\phi) = \frac{6m}{24m \wp(\omega_0 \mp \sqrt{2m} \phi; g_2, g_3) + 1}, \quad (23)$$

where  $\wp(x; g_2, g_3)$  is the  $\wp$ -Weierstrass elliptic function, with the Weierstrass invariants given by

$$g_2 = \frac{1}{48m^2} - \frac{\tilde{\mu}^2}{4h^2}, \quad g_3 = \frac{1}{1728m^3} - \frac{\tilde{\mu}^2}{96mh^2} - \frac{k^2 - 2\tilde{\mu}^2}{64mh^2}, \quad \text{and} \quad \omega_0 = \wp^{-1} \left[ \frac{1}{4R_0} - \frac{1}{24m} \right], \quad (24)$$

being  $\omega_0$  an integration constant and  $R_0$  the return point of the particle.

### A. Bounded orbits

#### 1. Planetary orbit

Orbits of the first kind occur when the energy lies in the range  $V(r_S) < k^2/2 < V(r_U) < \tilde{\mu}^2$ , and this case require that  $P(r) = 0$  allows three real roots, all of which are positive; and we shall write them as

$$r_d^{(\nu)}(k, h) = \frac{1}{3} \left( \frac{2m\tilde{\mu}^2}{\tilde{\mu}^2 - k^2/2} \right) + \sqrt{\frac{\eta_2}{3}} \cos \left[ \frac{1}{3} \arccos \left( 3\eta_3 \sqrt{\frac{3}{\eta_2^3}} \right) + \frac{2\pi\nu}{3} \right], \quad (\nu = 0, 1, 2) \quad (25)$$

where

$$\eta_2 = 4 \left[ \frac{1}{3} \left( \frac{2m\tilde{\mu}^2}{\tilde{\mu}^2 - k^2/2} \right)^2 - \left( \frac{h^2}{\tilde{\mu}^2 - k^2/2} \right) \right], \quad (26)$$

$$\eta_3 = 4 \left[ \frac{2}{27} \left( \frac{2m\tilde{\mu}^2}{\tilde{\mu}^2 - k^2/2} \right)^3 - \frac{1}{3} \left( \frac{2m\tilde{\mu}^2}{\tilde{\mu}^2 - k^2/2} \right) \left( \frac{h^2}{\tilde{\mu}^2 - k^2/2} \right) + \frac{2mh^2}{\tilde{\mu}^2 - k^2/2} \right], \quad (27)$$



so, we can identify the apoastro distance as  $r_d^{(0)} = r_A$ , and the periastro distance as  $r_d^{(2)} = r_P$ , while the third solution can be recognized as the apoastro distance to the orbits of the second kind,  $r_F = r_d^{(1)}$ , see Fig. 2. In this way, we can rewrite the characteristic polynomial (22) as

$$P(r) = (r_A - r)(r - r_P)(r - r_F). \quad (28)$$

Furthermore, we can determine the angle of precession corresponding to an oscillation, resulting  $\Phi = 2\phi_P - 2\pi$ , where  $\phi_P$  is the angle from the apoastro to the periastro.

$$\Phi = \sqrt{\frac{2}{m}} \left( \wp^{-1} \left[ \frac{1}{4r_P} - \frac{1}{24m} \right] - \wp^{-1} \left[ \frac{1}{4r_A} - \frac{1}{24m} \right] \right) - 2\pi. \quad (29)$$

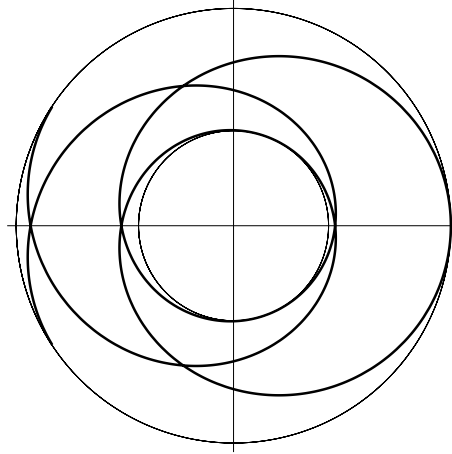


FIG. 2. Bounded orbit with precession for photons. Periastron  $r_P = 7.09$  and Apoastron  $r_A = 16.21$  for particles with  $m = 1$ ,  $h = 1.20$ ,  $k = 0.43$ , and  $\tilde{\mu}^2 = 0.1$ .

Also, If  $h = h_{LSO}$  the particle can orbit in a stable circular orbit at  $r_{LSO} = 6m$ . Besides, there is a critical orbit that approximates asymptotically to the stable circular orbit, see Fig. 3.

The second kind trajectory, the particle starts from a finite distance greater than the horizon, and it plunge towards the centre, see Fig. 4.

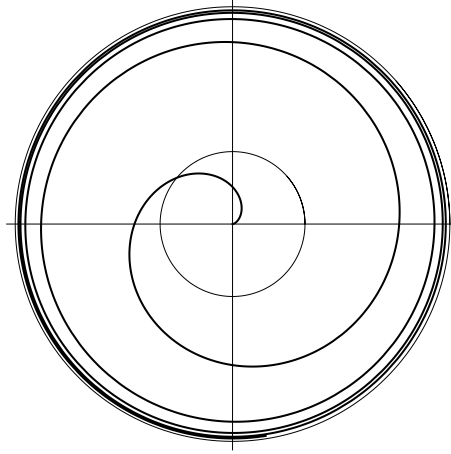


FIG. 3. The last critical orbit for photons. Apoastron  $r_{LSCO} = 6m$  for particles with  $m = 1$ ,  $h = 1.10$ ,  $k = 0.42$ , and  $\tilde{\mu}^2 = 0.1$ .

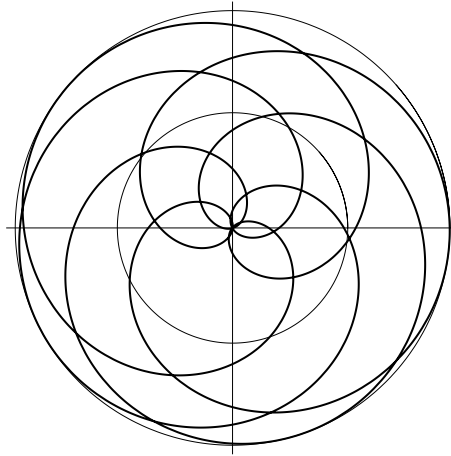


FIG. 4. Bounded orbit of second kind for photons. The return point is 3.77 for particles with  $m = 1$ ,  $h = 1$ ,  $k = 0.4$ , and  $\tilde{\mu}^2 = 0.1$ .

## B. Unbounded orbits

### 1. Deflection of light

Orbits of the first kind occur when the energy lies in the range  $\tilde{\mu}^2 < k^2/2 < V(r_U)$ , and this case requires that  $P(r) = 0$  allows three real roots, which we can identify as  $r_d^{(0)} = r_D$  that corresponds to the closest distance,  $r_d^{(2)} = r_f$  as a apoastro distance for the trajectories of second kind and the third root,  $r_d^{(1)} = r_3$ , is negative without physical interest. In this

way, we can rewrite the characteristic polynomial (22) as

$$P(r) = (r - r_D)(r - r_f)(r - r_3). \quad (30)$$

Furthermore, we can determine the angle of scattering, resulting  $\Theta = 2\phi_\infty - \pi$

$$\Theta = \sqrt{\frac{2}{m}} \left( \wp^{-1} \left[ \frac{1}{4r_D} - \frac{1}{24m} \right] - \wp^{-1} \left[ -\frac{1}{24m} \right] \right) - \pi, \quad (31)$$

where  $\phi_\infty$  is the angle from the closest distance to the infinity, see Fig. 5.

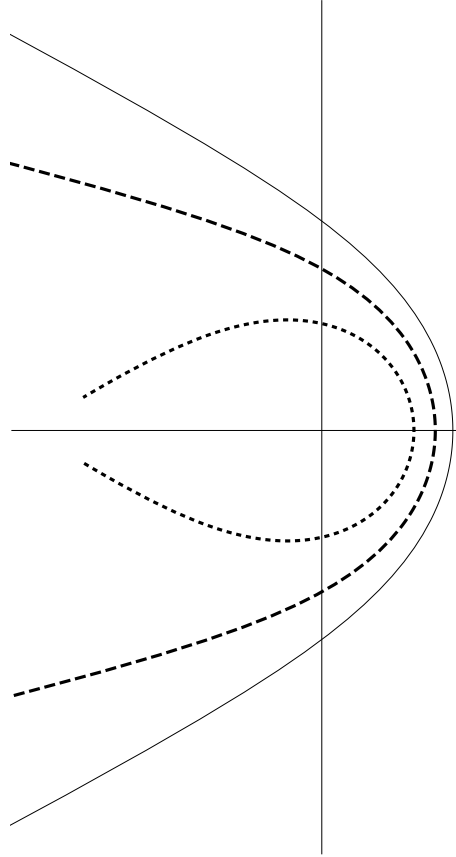


FIG. 5. Deflection of light with  $m = 1$ ,  $\tilde{\mu}^2 = 0.1$ ,  $h = 2$ ,  $k = 0.55$  (Thin line),  $k = 0.5725$  (dashed line), and  $k = 0.6$  (dotted line).

Also, there are orbits where the particle can scape to the infinity or plunge into the horizon, see Fig. 6. On the other hand, there are two critic orbits that approximate asymptotically to the unstable circular orbit. First kind, the particle arises from the infinity, and the second kind, the particle starts from a finite distance greater than the horizon, but smaller than the unstable radius, see Fig. 7.

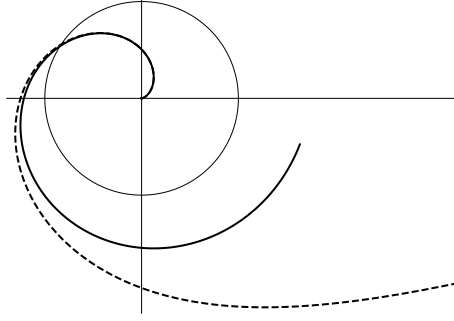


FIG. 6. Unbounded trajectories for photons with  $m = 1$ ,  $\tilde{\mu}^2 = 0.1$ ,  $h = 2$ ,  $k = 0.61$  (continuous line), and  $k = 0.65$  (dashed line).

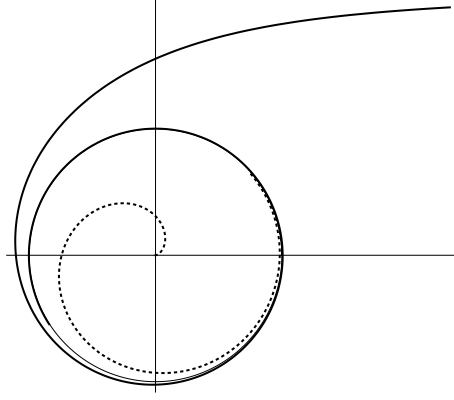


FIG. 7. Critical trajectories for photons with  $m = 1$ ,  $\tilde{\mu}^2 = 0.1$ ,  $h = 2$ , and  $k = 0.61$ . First kind (continuous line) and second kind (dotted line)

## V. RADIAL TRAJECTORIES

Radial motion corresponds to a trajectory with vanished angular momentum. So, the effective potential for the particle is

$$V(r) = \tilde{\mu}^2 \left( 1 - \frac{2m}{r} \right). \quad (32)$$

Observe that the potential for photons does not vanish as the Schwarzschild case, where the potential is null for photons. This implies that there are bounded radial trajectories as we will see.

### A. Bounded trajectories:

It is possible to observe bounded trajectories if the condition  $k^2/2 < \tilde{\mu}^2$  is satisfied. The return point is given by

$$r_0 = \frac{2m\tilde{\mu}^2}{\tilde{\mu}^2 - k^2/2}. \quad (33)$$

$$(34)$$

The particles all plunge into the horizon and the proper time  $\tau(r)$  solution yields

$$\tau(r) = \sqrt{\frac{r_0}{2m\tilde{\mu}^2}} \left[ r_0 \arctan \sqrt{\frac{r_0}{r} - 1} + r \sqrt{\frac{r_0}{r} - 1} \right], \quad (35)$$

$x^4(r, b)$  reads

$$x^4(r, b) = \left( \frac{k}{2} + \frac{b}{2} \right) \tau(r) + \frac{k}{2} \sqrt{\frac{2mr_0}{\tilde{\mu}^2}} \left[ 2 \arctan \sqrt{\frac{r_0}{r} - 1} + \sqrt{\frac{2m}{r_0 - 2m}} \ln \Omega(r) \right], \quad (36)$$

where

$$\Omega(r) = \left| \frac{\sqrt{r(r_0 - 2m)} + \sqrt{2m(r_0 - r)}}{\sqrt{r(r_0 - 2m)} - \sqrt{2m(r_0 - r)}} \right| \quad (37)$$

and  $x^5(r, b) = x^4(r, -b)$ . In Fig. 8, we plot the behavior of the proper time  $\tau$  and the temporal coordinate  $\tilde{x}^4 = (x^4 + x^5)/\sqrt{2}$ , where the particle crosses the horizon in a finite proper time and the particle takes an infinity time to reach the horizon.

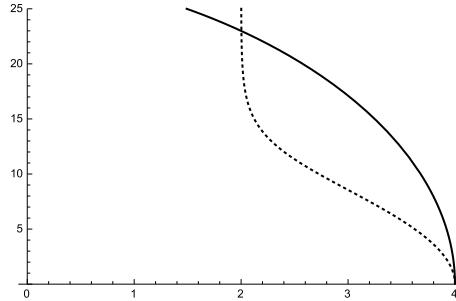


FIG. 8. Bounded radial trajectories for photons with  $m = 1$ ,  $\tilde{\mu}^2 = 0.1$ , and  $r_0 = 4$ . Continuous line for the proper time  $\tau$  and dotted line for the temporal coordinate  $\tilde{x}^4$ .

## B. Unbounded trajectories:

Also, there are unbounded trajectories if the condition  $k^2/2 \geq \tilde{\mu}^2$  is satisfied. In this case the proper time  $\tau(r)$ ,  $x^4(r, b)$  and  $x^5(r, b)$  yield

$$\tau(r) = \pm \frac{2}{3\sqrt{2m\tilde{\mu}^2}} \left( r^{3/2} - r_0^{3/2} \right) \quad (38)$$

$$x^4(r, b) = \left( \frac{k}{2} + \frac{b}{2} \right) \tau(r) \pm \frac{k}{2} \sqrt{\frac{2m}{\tilde{\mu}^2}} \left[ 2(\sqrt{r} - \sqrt{r_0}) - \sqrt{2m} \ln \left| \frac{\sqrt{r} + \sqrt{2m}}{\sqrt{r} - \sqrt{2m}} \frac{\sqrt{r_0} - \sqrt{2m}}{\sqrt{r_0} + \sqrt{2m}} \right| \right]$$

$$x^5(r, b) = x^4(r, -b),$$

where, by simplicity, we have considered  $k^2/2 = \tilde{\mu}^2$ . In Fig. 9, we plot the behavior of the proper time  $\tau$  and the temporal coordinate  $\tilde{x}^4$ , where the particle crosses the horizon in a finite proper time and the particle takes an infinity time to reach the horizon.

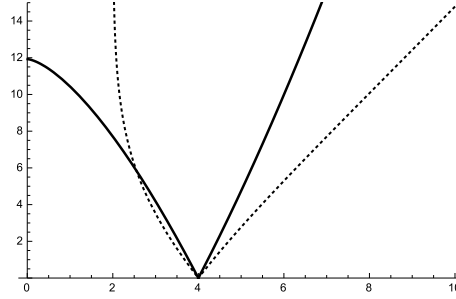


FIG. 9. Unbounded radial trajectories for photons with  $m = 1$ ,  $\tilde{\mu}^2 = 0.1$ , and  $r_0 = 4$ . Continuous line for the proper time  $\tau$  and dotted line for the temporal coordinate  $\tilde{x}^4$ .

## VI. SUMMARY

In this manuscript, we have studied the geodesic structure for a geometry described by a spherically symmetric four-dimensional solution embedded into a five-dimensional space known as brane-based spherically symmetric solution, and we have described the different kinds of orbits for particles and photons. Mainly, we have found that the extra dimension contributes to the particle test mass, that we have called *the anomalous test mass* which is positive for photons. This fact implies that there are bounded orbits for the photons, and

we have found a stable circular orbit and the associated epicyclic frequency. Also, we have found bounded orbits that oscillate between an apoastro and a periastro distance, and we have determined the perihelion shift for the photons that have not been observed for other geometries. In addition, we have found that the deflection of light is allowed for values of the energy greater than the anomalous test mass and less than the energy for the unstable circular orbit. Note that this fact does not occur for deflection of light in a Schwarzschild spacetime [24]. The geometry analyzed presents bounded and unbounded radial trajectories for neutral particles. However, it is possible to find bounded trajectories for photons as a new behavior observed for null geodesics. In this sense, bounded orbits for the photons can be seen as a consequence of extra dimensions.

## ACKNOWLEDGMENTS

This work was partially funded by Comisión Nacional de Ciencias y Tecnología through FONDECYT Grant 11140674 (PAG). P. A. G. acknowledges the hospitality of the Universidad de La Serena and Pontificia Universidad Católica de Valparaíso, where part of this work was undertaken.

- 
- [1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys. ) **1921**, 966 (1921).
  - [2] A. Font and S. Theisen, Lect. Notes Phys. **668**, 101 (2005).
  - [3] C. S. Chan, P. L. Paul and H. L. Verlinde, Nucl. Phys. B **581**, 156 (2000) [hep-th/0003236].
  - [4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [hep-ph/9905221].
  - [5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [hep-th/9906064].
  - [6] R. Maartens, Living Rev. Rel. **7**, 7 (2004) [gr-qc/0312059].
  - [7] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485**, 208 (2000) [hep-th/0005016].
  - [8] S. del Campo and R. Herrera, Phys. Lett. B **653**, 122 (2007) [arXiv:0708.1460 [gr-qc]].
  - [9] R. Gregory, N. Kaloper, R. C. Myers and A. Padilla, JHEP **0710**, 069 (2007) [arXiv:0707.2666 [hep-th]].
  - [10] A. Lue, Phys. Rept. **423**, 1 (2006) [astro-ph/0510068].

- [11] G. R. Dvali, G. Gabadadze, M. Kolanovic and F. Nitti, Phys. Rev. D **64**, 084004 (2001) [hep-ph/0102216].
- [12] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002) [astro-ph/0105068].
- [13] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, Phys. Rev. D **65**, 044026 (2002) [hep-th/0106001].
- [14] J. Saavedra and Y. Vasquez, JCAP **0904**, 013 (2009) doi:10.1088/1475-7516/2009/04/013 [arXiv:0803.1823 [gr-qc]].
- [15] K. i. Maeda, S. Mizuno and T. Torii, Phys. Rev. D **68**, 024033 (2003) [gr-qc/0303039].
- [16] R. Emparan, R. Gregory and C. Santos, Phys. Rev. D **63**, 104022 (2001) [hep-th/0012100].
- [17] A. Chamblin, S. W. Hawking and H. S. Reall, Phys. Rev. D **61** (2000) 065007 [hep-th/9909205].
- [18] V. P. Frolov and A. Zelnikov and U. Bleyer, Ann. Phys. (Leipzig) **44**, 371 (1987).
- [19] A. N. Aliev and G. D. Esmer, Phys. Rev. D **87** (2013) 8, 084022
- [20] R. R. Cuzinatto, P. J. Pompeia, M. Montigny, F. C. Khanna and J. M. H. da Silva, Eur. Phys. J. C **74** (2014) 8, 3017 [arXiv:1405.0526 [gr-qc]].
- [21] C. G. Boehmer, G. De Risi, T. Harko and F. S. N. Lobo, Class. Quant. Grav. **27** (2010) 185013 [arXiv:0910.3800 [gr-qc]].
- [22] G. Alencar, C. R. Muniz, R. R. Landim, I. C. Jardim and R. N. C. Filho, arXiv:1511.03608 [hep-th].
- [23] J. Ramos-Caro, J. F. Pedraza and P. S. Letelier, Mon. Not. Roy. Astron. Soc. **414**, 3105 (2011) [arXiv:1103.4616 [astro-ph.EP]].
- [24] Chandrasekhar S.: The Mathematical Theory of Black Holes. Oxford University Press, New York (1983).